A universal adiabatic quantum query algorithm

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Based on joint work with

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[BrandehoR, TQC'15, arxiv:1409.3558]

Outline

* Preliminaries
  * Adiabatic quantum computation
  * Quantum query complexity
* Continuous-time quantum query complexity
  * Lower bound: adversary bound
  * Upper bound: adiabatic algorithm
* Conclusion and discussion

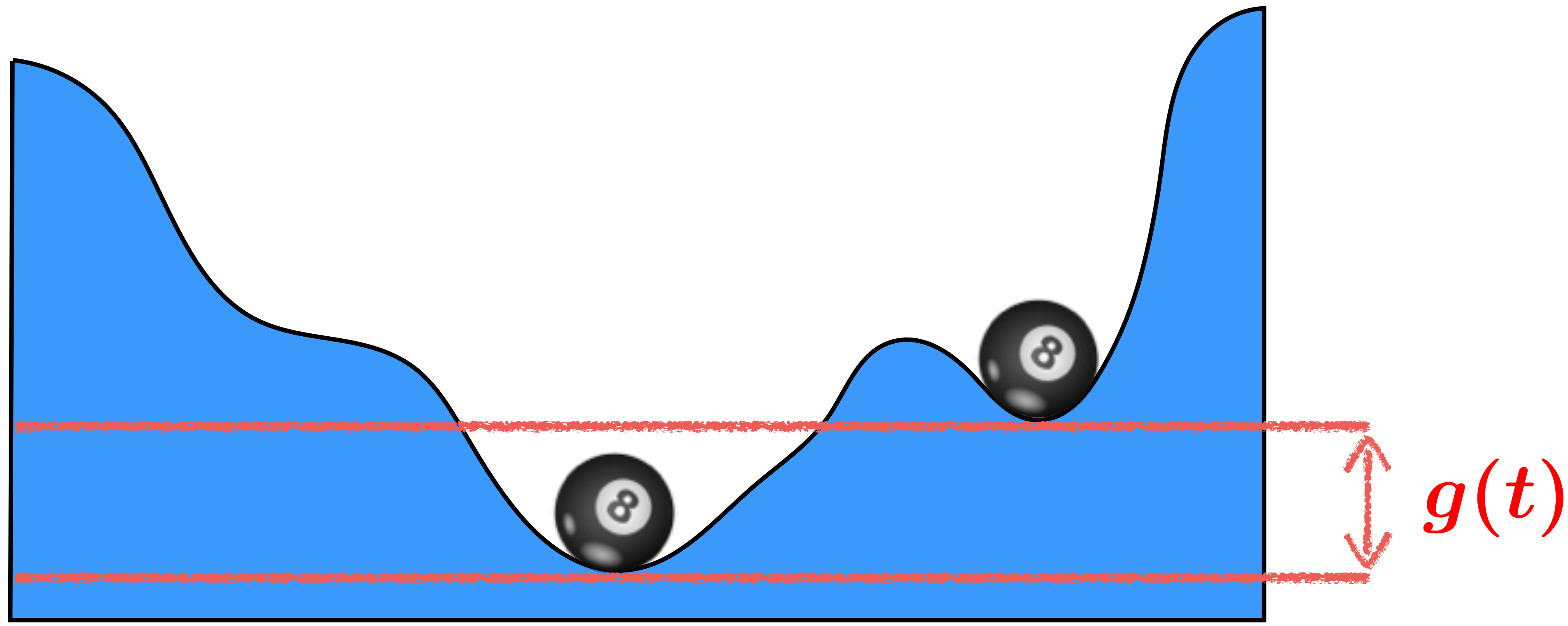
Adiabatic quantum computation

Discrete vs continuous-time quantum computation

	Discrete	Continuous
Building blocks	2 qubit gates U_i	2-local Hamiltonians h_i
Algorithm	Sequence of gates $U_T \dots U_2 U_1$	Hamiltonian $H = \sum_i h_i$
Complexity	Total number of gates	Total time of evolution under H

polynomially equivalent

Adiabatic evolution



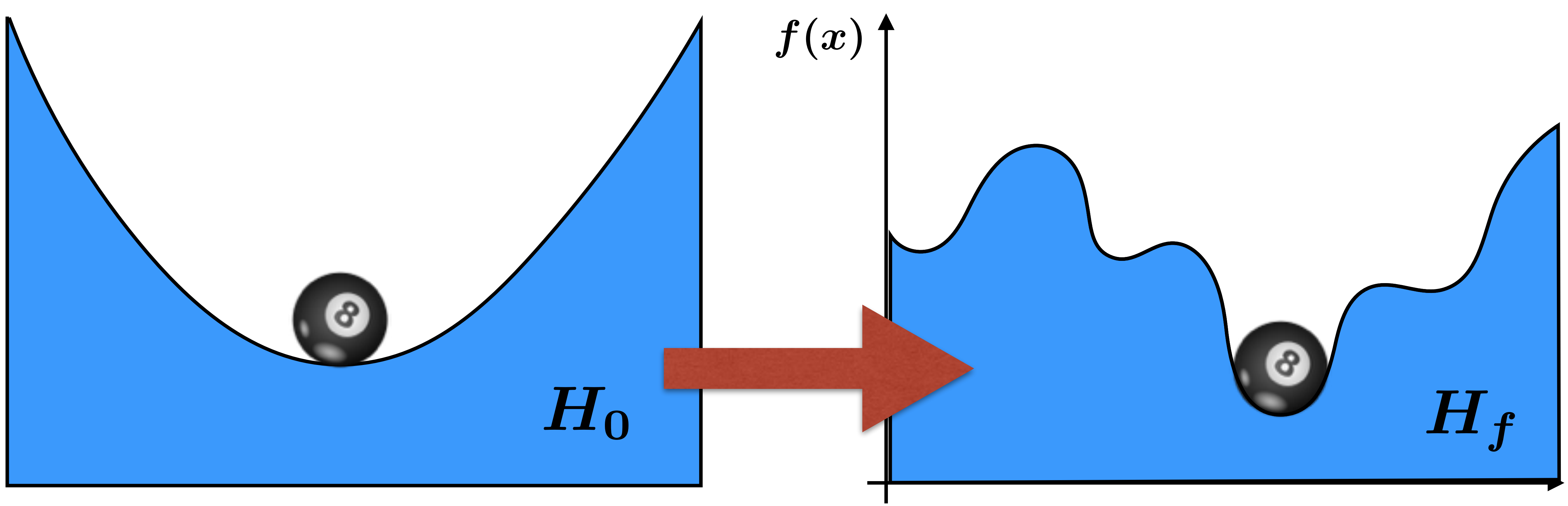
- Slow evolution $\Longrightarrow$ remains in ground state
- Probability of excitation depends on
  - Total time T (slower is better)
  - Gap $g(t)$ (larger is better)

$$\boxed{T \gg \frac{1}{g_{\min}^3}}$$

Adiabatic quantum computation

[Farhi *et al.*'00]

- Problem: find minimum of function $f(x)$
  - Prepare ground state of simple Hamiltonian H_0
  - Slowly switch to H_f with spectrum matching $f(x)$



How powerful is it?

- It is quantum
  - Unstructured search in time $O(\sqrt{N})$ (cf Grover)

[vanDam-Mosca-Vazirani'01, RCerf'02]

- It is universal for quantum computation [Aharonov *et al.* '05]
- Initial motivation: optimization problems (NP-complete)
  - Worst case: exponential [vanDam-Vazirani'03,Reichardt'04]
  - Average case: long debate (numerical simulations)

Note

- Few known adiabatic algorithms
- Mostly heuristics (no analytical results)

Quantum query complexity

Classical query complexity

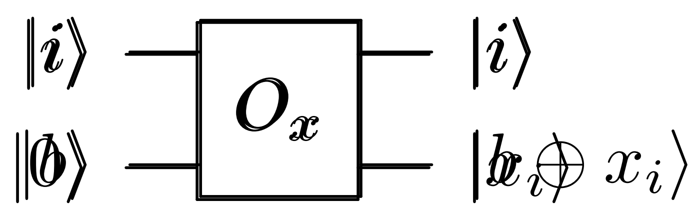
- Function $f(x)$, where $x = (x_1, \ldots, x_n)$
- Oracle $O_x : i \to x_i$
- Goal: Compute $f(x)$ given black-box access to O_x

Randomized query complexity $R_\varepsilon(f)$

Minimum # calls to O_x necessary to compute $f(x)$ with success probability $(1 - \varepsilon)$

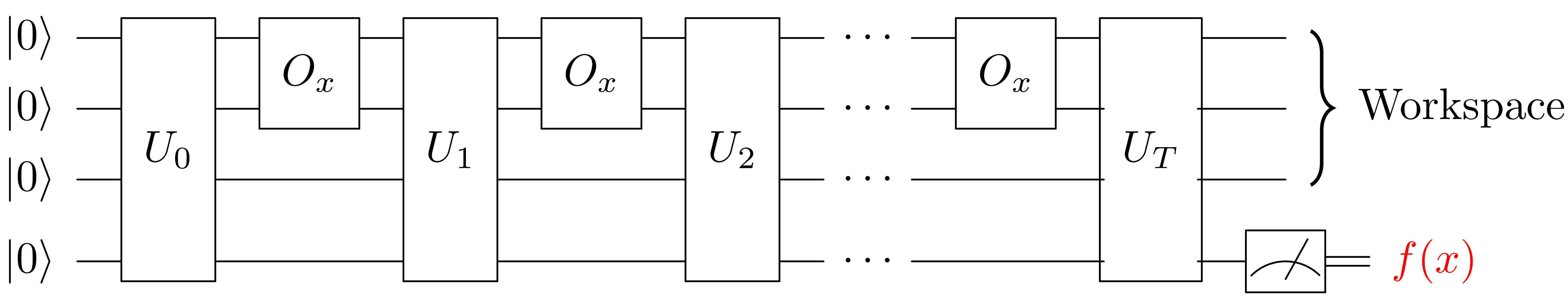
Quantum query complexity

* Quantum oracle:



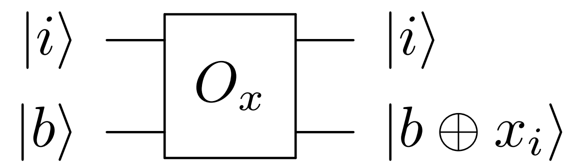
* Extra power:

Can query O_x in superposition $\Rightarrow Q_\varepsilon(f) \leq R_\varepsilon(f)$



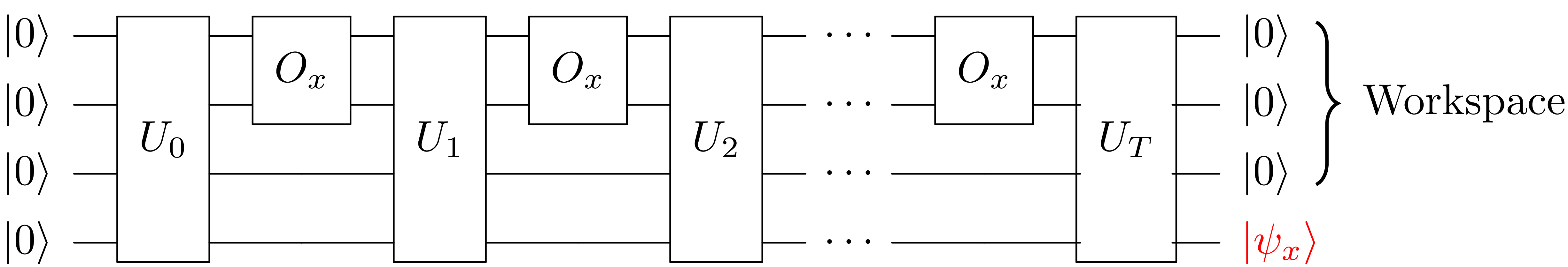
Quantum state generation

- Set of quantum states $\{|\psi_x\rangle : x \in \mathcal{D}^n\}$
- Goal: Generate $|\psi_x\rangle$ given black-box access to O_x



- Observation: Problem only depends on Gram matrix

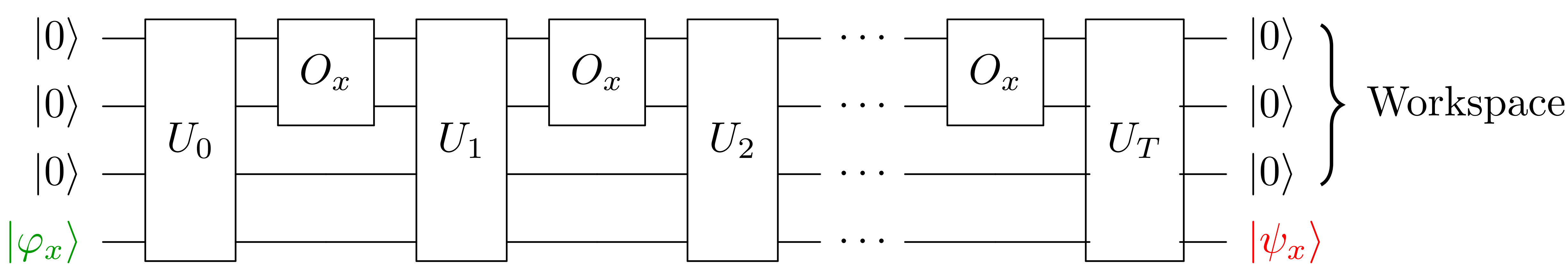
$$M_{xy} = \langle \psi_x | \psi_y \rangle$$



Quantum state conversion

- Set of target states $\{|\psi_x\rangle : x \in \mathcal{D}^n\}$
- Set of initial states $\{|\varphi_x\rangle : x \in \mathcal{D}^n\}$
- Goal: Convert $|\varphi_x\rangle$ to $|\psi_x\rangle$ given black-box access to O_x
- Observation: Problem only depends on Gram matrices

$$M_{xy} = \langle \psi_x | \psi_y \rangle \qquad N_{xy} = \langle \varphi_x | \varphi_y \rangle$$



(Zero-error) quantum query complexity

- Given
  - Gram matrix of initial states N
  - Gram matrix of target states M
  - Black-box access to x via oracle O_x

Quantum query complexity $Q_0(N, M)$

Minimum # calls to O_x necessary to convert the state $|\varphi_x\rangle|\bar{0}\rangle$ into $|\psi_x\rangle|\bar{0}\rangle$

work space

Bounded-error quantum query complexity

- Given
  - Gram matrix of initial states N
  - Gram matrix of target states M
  - Black-box access to x via oracle O_x

Quantum query complexity $Q_\varepsilon(N, M)$

Minimum # calls to O_x necessary to convert the state $|\varphi_x\rangle|\bar{0}\rangle$ into a state

$$\sqrt{1-\varepsilon}|\psi_x\rangle|\bar{0}\rangle + \sqrt{\varepsilon}|\mathrm{error}_x\rangle$$

Reducing to zero-error case

- $|\psi_x^t\rangle$: state of the algorithm after t queries on input x
- Gram matrix $M_{xy}^t = \langle\psi_x^t|\psi_y^t\rangle$
- Initially: $|\psi_x^0\rangle = |\varphi_x\rangle|\bar{0}\rangle \Rightarrow M^0 = N$
- At the end: $|\psi_x^T\rangle \approx |\psi_x\rangle|\bar{0}\rangle \Rightarrow M^T \approx M$

What distance?

Output conditions

* $\|M^T - M\|_\infty \leq 2\sqrt{\varepsilon}$ [Ambainis02]
* $\gamma_2(M^T - M) \leq 2\sqrt{\varepsilon}$ [HøyerLeeŠpalek07]
* $\mathcal{F}_H(M^T, M) \geq \sqrt{1-\varepsilon}$ [LeeR11]

where $\mathcal{F}_H(M^T, M) = \min_{|u\rangle} \mathcal{F}(M^T \circ |u\rangle\langle u|, M \circ |u\rangle\langle u|)$

- Theorem: The last condition is tight

$$\boxed{Q_\varepsilon(N, M) = \min_{\mathcal{F}_H(M,M') \geq \sqrt{1-\varepsilon}} Q_0(N, M')}$$

Quantum lower bounds

* Different lower bound methods :
  - Adversary method:
    - Idea: bound the change in a progress function for each query
  - Polynomial method:
    - Idea: bound the degree of polynomials approximating the function

Adversary bound

[HøyerLeeŠpalek07]

- Progress function: $\mathcal{W}[M^t] = \mathbf{Tr}[(\mathbf{\Gamma} \circ M^t)vv^*]$
- Initial value: $\mathcal{W}[N] = \mathbf{Tr}[(\mathbf{\Gamma} \circ N)vv^*]$
- Additive change for one query:

Adversary matrix

$$\|\mathbf{\Gamma} \circ \mathbf{\Delta}_i\| \leq 1 \ \forall i \ \Rightarrow \ \left|\mathcal{W}[M^{t+1}] - \mathcal{W}[M^t]\right| \leq 1$$

- Final value after T queries: $|\mathcal{W}[M^T] - \mathcal{W}[M^0]| \leq T$

Adversary bound

$$\mathrm{ADV}(N, M) = \max_{\mathbf{\Gamma}} \|\mathbf{\Gamma} \circ (M - N)\|$$

subject to $\|\mathbf{\Gamma} \circ \mathbf{\Delta}_i\| \leq 1 \ \forall i$

Adversary bound is tight

- In the bounded-error case, we have:
  - $\mathrm{ADV}_\varepsilon(f)$ is a lower bound for $Q_\varepsilon(f)$ [HøyerLeeŠpalek'07]
  - $\mathrm{ADV}_\varepsilon(f)$ is also an upper bound! [Reichardt'11,LMRŠS'11]
- Proof idea:
  - $\mathrm{ADV}_\varepsilon(f)$ can be expressed as a semidefinite program (SDP)
  - Dualize this SDP
  - Build an algorithm from a feasible point of the dual SDP

Continuous-time quantum query complexity

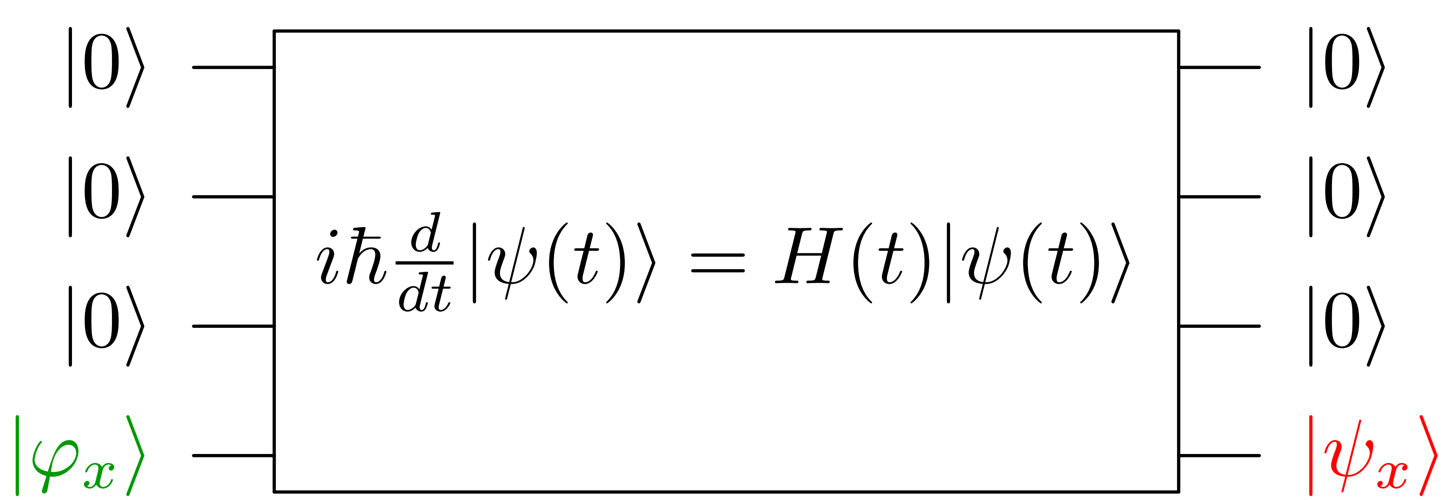
Continuous-time state conversion

- Set of target states $\{|\psi_x\rangle : x \in \mathcal{D}^n\}$
- Set of initial states $\{|\varphi_x\rangle : x \in \mathcal{D}^n\}$
- Given Hamiltonian oracle H_x (s.t. $O_x = e^{-iH_x}$)
- Convert $|\varphi_x\rangle$ to $|\psi_x\rangle$ via evolution under

$$H(t) = H_D(t) + \alpha(t) H_x$$

arbitrary

$|\alpha(t)| \leq 1$



Continuous-time quantum query complexity

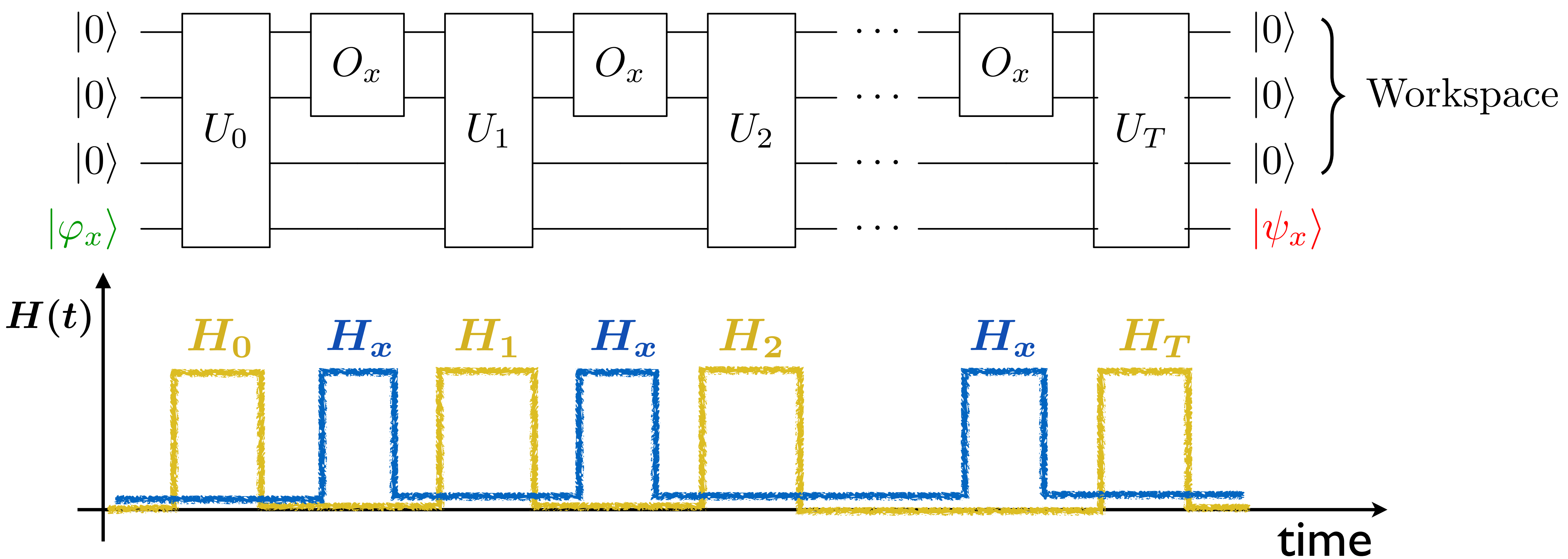
- Given
  - Gram matrix of initial states N
  - Gram matrix of target states M
  - Black-box access to x via Hamiltonian oracle H_x

C-t quantum query complexity $Q_0^{\mathrm{ct}}(N, M)$

Minimum time of evolution under $H(t) = H_D(t) + \alpha(t) H_x$ necessary to convert the state $|\varphi_x\rangle|\bar{0}\rangle$ into $|\psi_x\rangle|\bar{0}\rangle$

Comparison with discrete-time model (1)

- Hamiltonian simulation of quantum circuit



$$\boxed{Q_0^{\mathrm{ct}}(N, M) \leq Q_0(N, M)}$$

Comparison with discrete-time model (2)

- Just as in the discrete-time case, we can prove that

$$\boxed{Q_0^{\text{ct}}(N,M) \geq \mathbf{ADV}(N,M)}$$

- Two proof approaches:
  - Adapting the discrete-time proof [Yonge-Mallo'11]
  - Reduction via the fractional query model [CGMSY'09,LMRŠS'11]

$$\boxed{Q_\varepsilon^{\text{ct}}(f) = \Theta(Q_\varepsilon(f)) = \Theta(\mathbf{ADV}(f))}$$

Our contribution

- We revisit this result
- For the lower bound
  - Direct proof
- For the upper bound
  - Adiabatic algorithm (inherently time-continuous)
- Motivation
  - New intuition
  - New ideas to build adiabatic quantum algorithms?

Lower bound

Continuous-time adversary bound

- Let $|\psi_x(t)\rangle$ be the state of the algorithm on input x at time t
- Assume we run the algorithm on a superposition of inputs

$$|\Psi(t)\rangle = \sum_x v_x |x\rangle_{\mathcal{X}} |\psi_x(t)\rangle_{\mathcal{A}}$$

- Choose an observable Γ on $\mathcal{X}$ measuring "progress"

$$\mathcal{W}(t) = \langle\Gamma\rangle_t = \langle\Psi(t)|\Gamma \otimes I_{\mathcal{A}}|\Psi(t)\rangle$$

- Bound the progress over the course of the algorithm

$$\langle\Gamma\rangle_T - \langle\Gamma\rangle_0 = \int_0^T \partial_t \langle\Gamma\rangle_t dt \leq T\,|\partial_t \langle\Gamma\rangle_t|$$


EHRENFESTWEG
PAUL EHRENFEST,1880-1933,NATUURKUNDIGE

Continuous-time adversary bound

$$\langle \Gamma \rangle_T - \langle \Gamma \rangle_0 = \int_0^T \partial_t \langle \Gamma \rangle_t dt \leq T \left| \partial_t \langle \Gamma \rangle_t \right|$$

- By Ehrenfest's theorem: $\partial_t \langle \Gamma \rangle_t = -i \langle [H, \Gamma] \rangle_t + \langle \partial_t \Gamma \rangle_t$
  - $\langle \partial_t \Gamma \rangle_t = 0$
  - $H = I_{\mathcal{X}} \otimes H_D + \sum_x |x\rangle\langle x| \otimes H_x$

$$\Rightarrow [H, \Gamma] = \sum_x [|x\rangle\langle x| \otimes H_x, \Gamma]$$

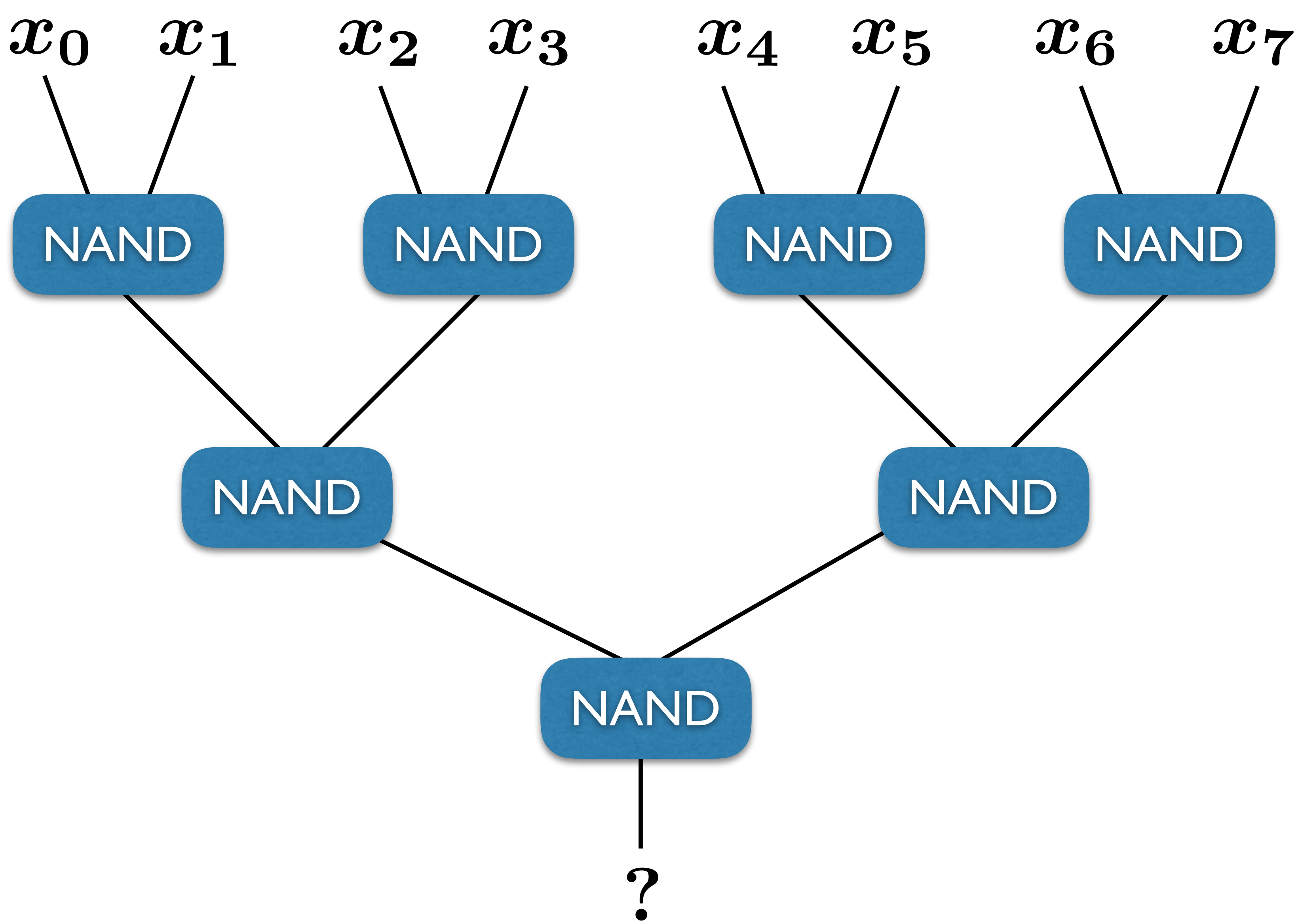
- We get the lower bound

$$T \geq \underbrace{\boxed{\max_{\Gamma} \left| \langle \Gamma \rangle_T - \langle \Gamma \rangle_0 \right| \text{ subject to } \| [H, \Gamma] \| \leq 1}}_{\mathrm{ADV}(N, M)}$$

Upper bound

NAND tree algorithm

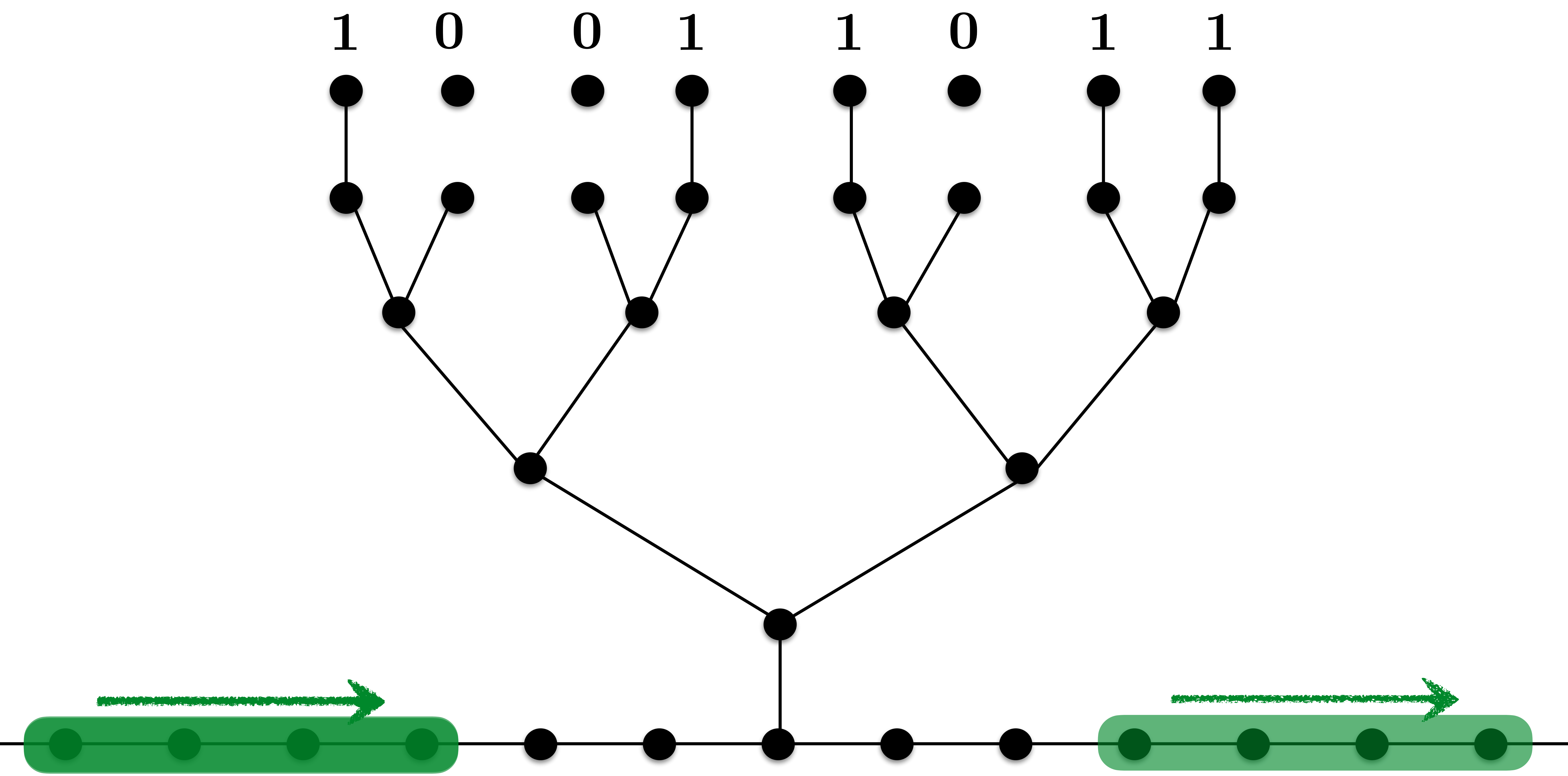
- Suppose we need to evaluate the following formula



- This can be done optimally (time $O(\sqrt{n})$) using a continuous-time quantum walk!

NAND tree algorithm

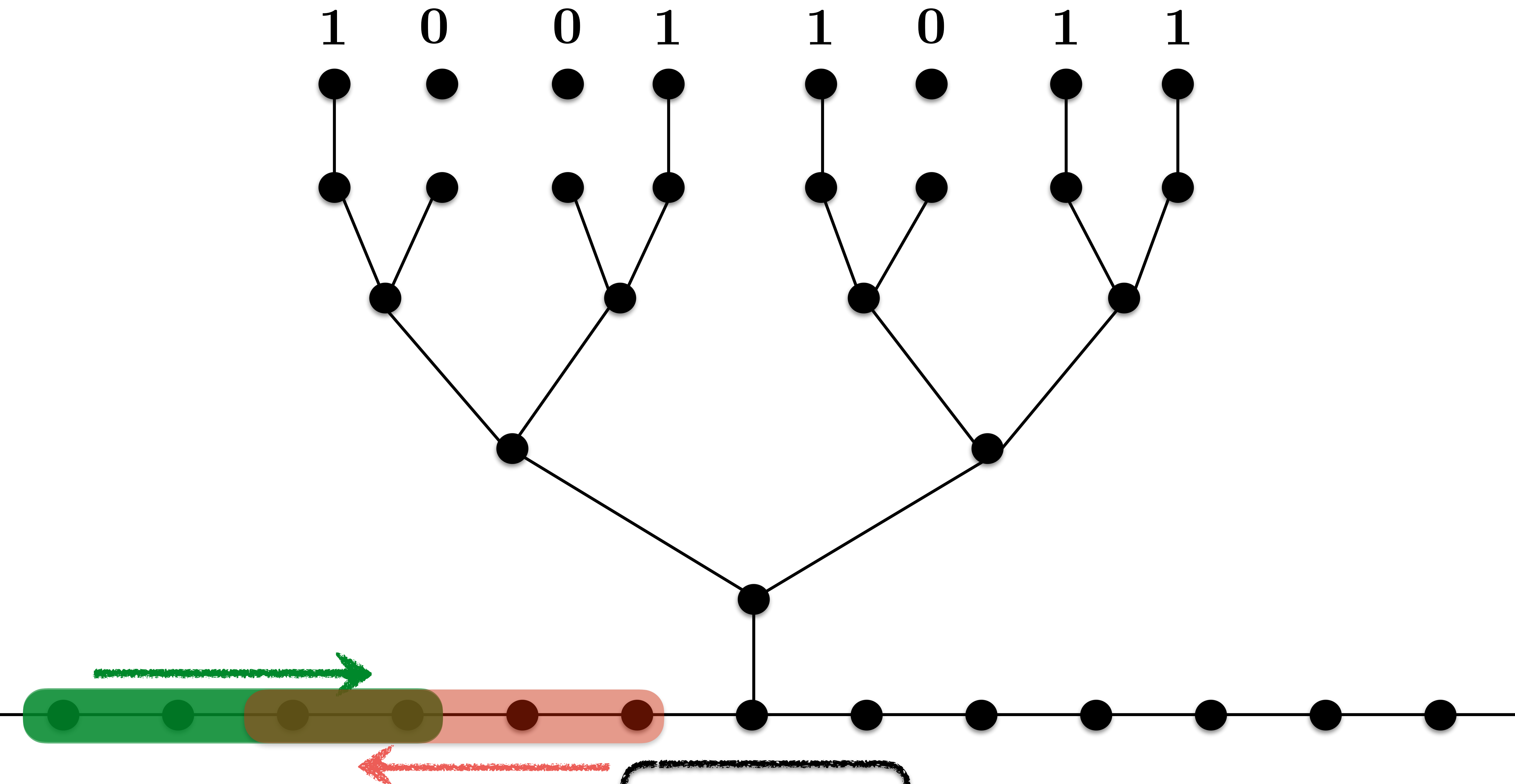
[Farhi-Goldstone-Gutmann'08]



$$\boxed{f(x) = 1}$$

NAND tree algorithm

[Farhi-Goldstone-Gutmann'08]



$$\boxed{f(x) = 0}$$

Dual of the adversary bound

$$\mathrm{ADV}(N,M) = \max_{\Gamma} \|\Gamma \circ (M-N)\|$$

subject to $\|\Gamma \circ \Delta_i\| \le 1 \ \forall i$

$\Updownarrow$ SDP dualization

$$\mathrm{ADV}(N,M) = \min_{|u_{x,j}\rangle, |v_{y,j}\rangle} \max\left\{ \max_x \sum_j \| |u_{x,j}\rangle \|^2, \max_y \sum_j \| |v_{y,j}\rangle \|^2 \right\}$$

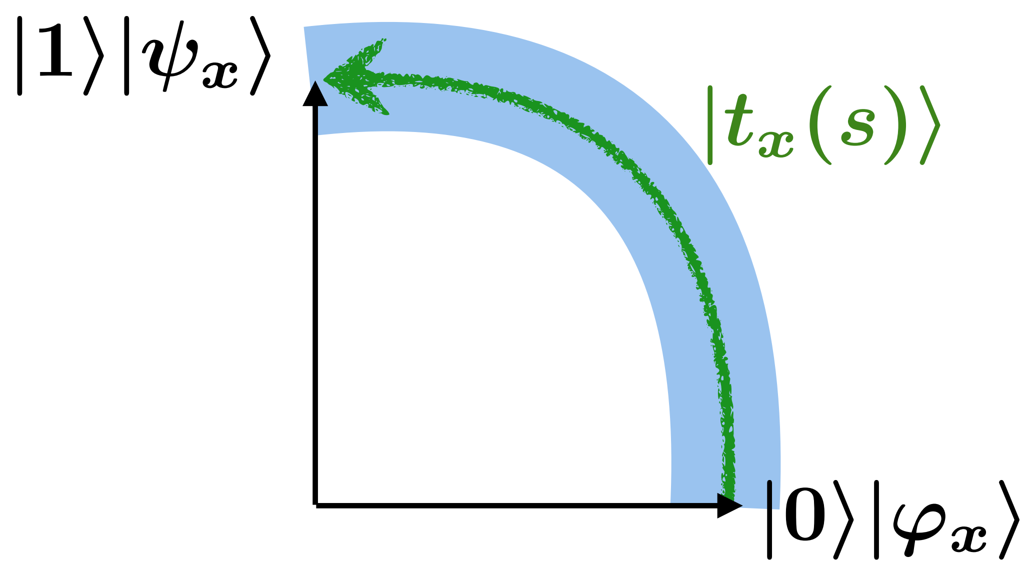
subject to

$$M_{xy} - N_{xy} = \sum_i \Delta_{i,xy} \langle u_{x,i} | v_{y,i} \rangle \ \forall x, y$$

Path to target state

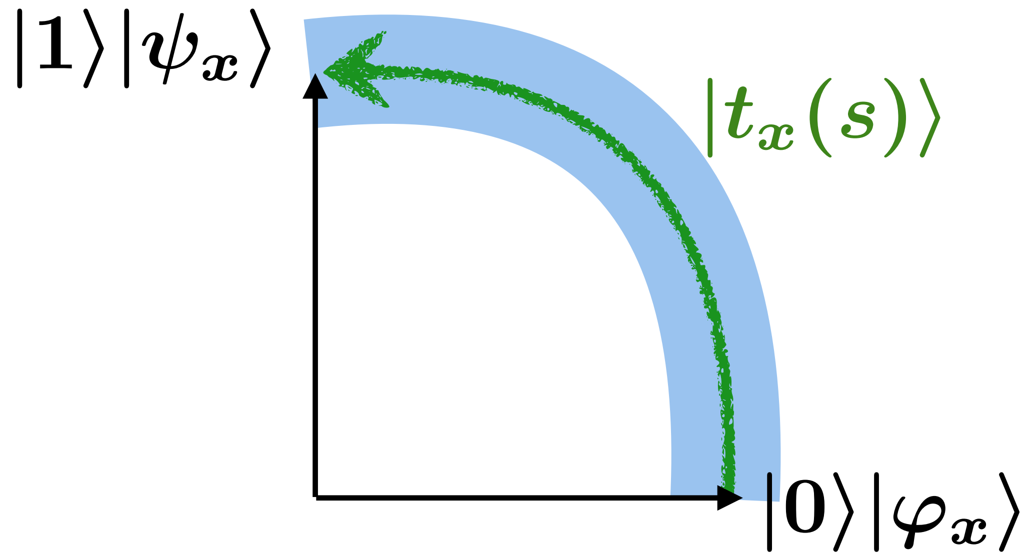
- Goal: convert $|\varphi_x\rangle$ to $|\psi_x\rangle$
- Ideal path: $|t_x(s)\rangle = \cos(\frac{\pi}{2}s)|0\rangle|\varphi_x\rangle + \sin(\frac{\pi}{2}s)|1\rangle|\psi_x\rangle$

$$(s = \frac{t}{T})$$



- Modified path $|\tilde{t}_x(s)\rangle = |t_x(s)\rangle + \frac{\delta}{\sqrt{\mathrm{ADV}(N,M)}}|u_x\rangle$
  - $|u_x\rangle$ built from $|u_{x,i}\rangle$ in dual form of $\mathrm{ADV}(N,M)$
  - $\left\| |\tilde{t}_x(s)\rangle - |t_x(s)\rangle \right\| \le \delta$

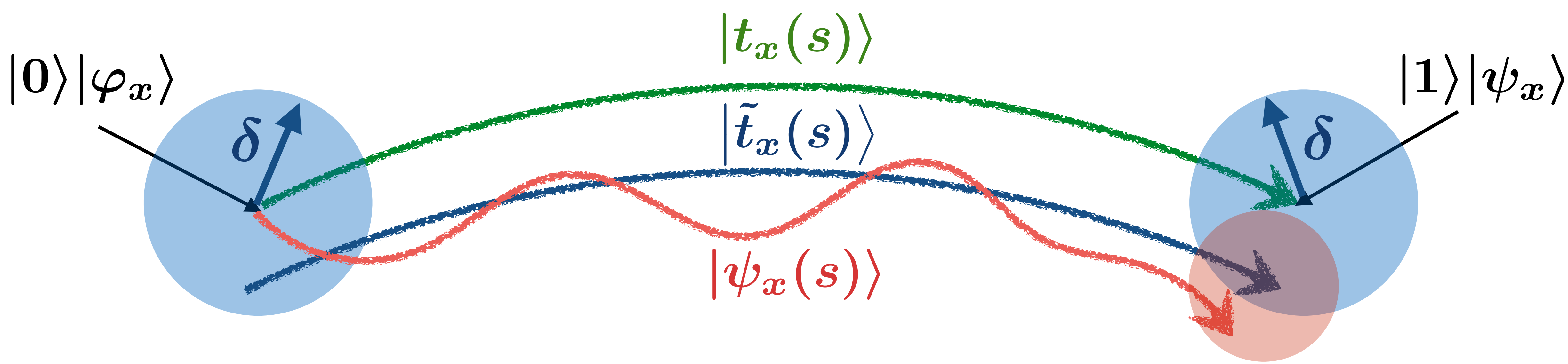
Hamiltonian



* We set $H(s) = \Pi(s) - H_x$ with $s = \frac{t}{T}$
  - Oracle Hamiltonian H_x
  - Driver Hamiltonian $\Pi(s)$: projector built from $|v_{x,i}\rangle$ in dual form of $\mathrm{ADV}(N, M)$

$$\boxed{H(s)\big|\tilde{t}_x(s)\big\rangle = 0 \quad \forall s}$$

Correctness of the algorithm



- Error analysis

$$\||\psi_x(1)\rangle - |1\rangle|\psi_x\rangle\| \leq \quad \text{starting error} \qquad \leq \delta$$
$$+ \text{ adiabatic error} \qquad \leq \varepsilon_A \; ?$$
$$+ \text{ ending error} \qquad \leq \delta$$

Adiabatic condition(s)

Let $g(s)$ be the spectral gap. Then

$$\varepsilon_A \leq \frac{1}{T} \max_s \left[2 \frac{||\dot{H}(s)||}{g^2(s)} + \frac{||\ddot{H}(s)||^2}{g^2(s)} + 7 \frac{||\dot{H}(s)||^2}{g^3(s)} \right]$$

[Jansen-Ruskai-Seiler'07]

Problem

Here, we might not have a gap!

Adiabatic condition(s)

Let $P(s) = |\tilde{t}_x(s)\rangle\langle\tilde{t}_x(s)|$ and $A(s)$ be such that

$$[\dot{P}(s), P(s)] = [H(s), A(s)]$$

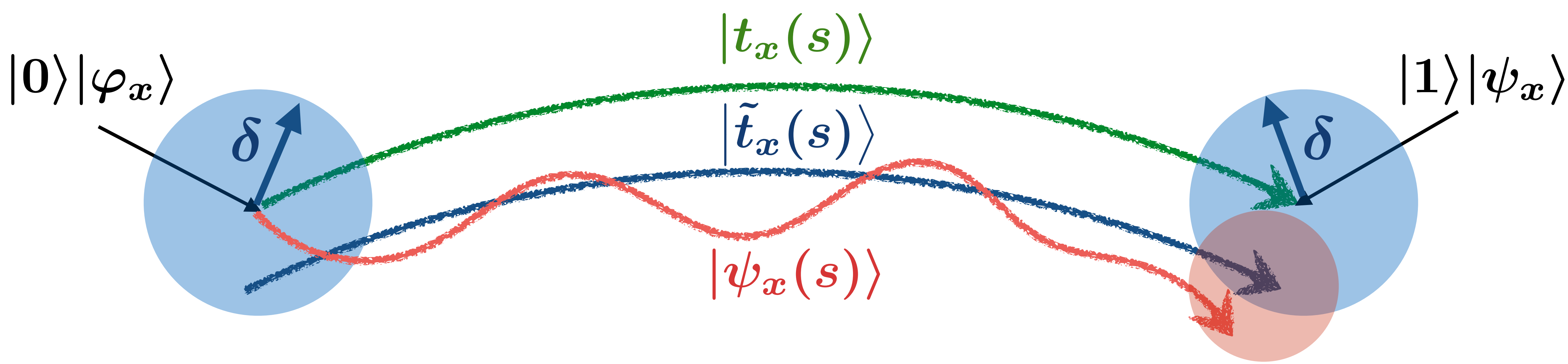
Then

$$\varepsilon_A \leq \frac{1}{T}\max_s \left[2||A(s)|| + ||\dot{A}(s)P(s)|| + ||A(s)\dot{P}(s)P(s)||\right]$$

[Avron-Elgart'99]

- Here: $A(s)$ built from $|v_{x,i}\rangle$ in dual form of $\mathrm{ADV}(N, M)$

Correctness of the algorithm



$$
\begin{aligned}
\| |\psi_x(1)\rangle - |1\rangle|\psi_x\rangle \| \leq \quad & \text{starting error} && \leq \delta \\
+ \; & \text{adiabatic error} && \leq 15\frac{\mathrm{ADV}(N,M)}{\delta T} \\
+ \; & \text{ending error} && \leq \delta
\end{aligned}
$$

* We choose running time $T = 15\frac{\mathrm{ADV}(N,M)}{\delta^2}$

$$\boxed{Q^{\mathrm{ct}}_{(3\delta)^2}(N,M) = O\left(\frac{\mathrm{ADV}(N,M)}{\delta^2}\right)}$$

Conclusion and discussion

Conclusion

- Alternative proof that the adversary bound characterizes $Q_\varepsilon^{\mathrm{ct}}$
  - Lower bound: Ehrenfest's theorem
  - Upper bound: Adiabatic condition without a gap
- New intuition:
  - Bounded error unavoidable due to adiabatic error

Further work

- Zero-error quantum query complexity
  - Non-adiabatic algorithm?
- New adiabatic quantum algorithms
  - Quantum query: adiabatic Deutsch-Jozsa, Simon, Shor?
  - Other: quantum walks?

Comparison with discrete-time adversary algorithm

	Continuous	Discrete
Technique	Adiabatic evolution	Phase estimation
Analysis	Adiabatic condition	Effective spectral gap lemma

Search via quantum walks

* Similar situation for quantum walks
  * Searching marked vertices from the stationary distribution (cf Maris' talk)

	Continuous	Discrete
Technique	Adiabatic evolution	Phase estimation
Analysis	Adiabatic condition	Effective spectral gap lemma

Search via quantum walks

* Similar situation for quantum walks
  * Detecting marked vertices from an arbitrary initial distribution (cf Alexander's talk)

	Continuous	Discrete
Technique	???	Phase estimation
Analysis	???	Effective spectral gap lemma

  * Can we also find multiple marked vertices using the adiabatic approach?